A definitional view of Vogt's variant of the Mazur-Ulam theorem

Victor Pambuccian

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1 Introduction

Perhaps the earliest result in the discipline that has come to be known as *characterizations of geometric transformations by weak hypotheses* is the theorem, proved by S. Mazur and S. Ulam [6], that surjective isometries between real normed spaces are affine transformations. It has been widely generalized, one of these generalizations being A. Vogt's [12] theorem that equidistance-preserving transformations betweeen real normed spaces of dimension ≥ 2 are affine similarities. F. Skof [10] has provided conditions for the conclusion of Vogt's theorem to hold without the surjectivity requirement.

The aim of this note is to rephrase the theorems of Vogt and Skof as theorems about the definability of a certain geometric notion in terms of another one. To be precise, whenever a theorem tells us that a map that preserves a certain relation ϱ must preserve another one ϱ' as well, we are told that ϱ' is implicitly definable in terms of ϱ. If the theorem can be phrased in a logical language that satisfies Beth's theorem, such as first-order logic or $\mathsf{L}_{\omega_1\omega}$, then we know that there must be an explicit definition of ϱ' in terms of ϱ. This fact is best expressed as (cf. [2, Th. 6.6.4, Ex. 6.6.2], [5], [3])

Preservation and Definability Theorem. *Let $\mathsf{L} \subseteq \mathsf{L}^+$ be two first-order or $\mathsf{L}_{\omega_1\omega}$ languages containing a sign for an identically false formula, $\mathcal{T}$ be a theory in L^+, and $\varphi(\mathbf{X})$ be an L^+-formula in the free variables $\mathbf{X} = (X_1, \ldots, X_n)$. Then the following assertions are $\varphi(\mathbf{X})$ be an L^+-formula in the free variables $\mathbf{X} = (X_1, \ldots, X_n)$. Then the following assertions are equivalent:*

(i) there is an L-formula $\psi(\mathbf{X})$ (which is positive existentia l; positive existential, but negated equality is allowed; positive) such that $\mathcal{T} \vdash \varphi(\mathbf{X}) \leftrightarrow \psi(\mathbf{X})$;

(ii) for any $\mathfrak{A}, \mathfrak{B} \in Mod(\mathcal{T})$, and each L-isomorphism (L-homomorphism; L-monomorphism; L-epimorphism) $f : \mathfrak{A} \to \mathfrak{B}$, the following condition is satisfied:

if $\mathbf{c} \in \mathfrak{A}^{\mathfrak{n}}$ and $\mathfrak{A} \models \varphi(\mathbf{c})$, then $\mathfrak{B} \models \varphi(f(\mathbf{c}))$.

The theorems of Vogt and Skof are stated in the form (ii); the aim of our paper is to state them purely syntactically, as in (i), for we believe that the syntactic understanding sheds new light into the mechanics of these theorems.

2 The Axiomatic Set-Up

To apply the above theorem, we need to find a theory $\mathcal{T}$, among whose models are real normed linear spaces. Since we believe that the Archimedean axiom is needed[1] , we cannot express $\mathcal{T}$ inside first-order logic, but will have to work within $\mathsf{L}_{\omega_1\omega}$ (or within transitive closure logic, which is weaker than $\mathsf{L}_{\omega_1\omega}$, but not as easily readable[2]).
Our theorem will be weaker than VOGT's, since we require not only the preservation of equidistance, but the preservation of the negation of equidistance as well, i. e. the map f between two real normed vector spaces needs to satisfy not only $\|x-y\| = \|u-v\| \Rightarrow \|f(x)-f(y)\| = \|f(u)-f(v)\|$, but also the reverse implication, so we have $\Leftrightarrow$ instead of $\Rightarrow$. It is an open problem whether VOGT's theorem remains valid if we replace $\mathbb{R}$ with any Archimedean ordered field. Both VOGT's proof [12] and the proof of VOGT's theorem in [7, p. 626-627] use particular topological properties of real normed spaces, such as the fact that the complement of a sphere consists of two connected components, which are no longer valid in our setting.
The theory $\mathcal{T}$ we are looking for is axiomatized in a language $\mathsf{L}_{\omega_1\omega}$, where $\mathsf{L} := \mathsf{L}(B, \equiv)$, with individual variables to be interpreted as *points*, and two relation symbols, a ternary one B and a quaternary one $\equiv$, with with $B(abc)$ to be read as 'point b lies between a and c (being allowed to be equal to one or to both endpoints)', and $ab \equiv cd$ to be read as 'a is as distant from b as c is from d', or equivalently 'segment ab is congruent to segment cd'.
Let Δ be an $\mathsf{L}(B)$ axiom system for ordered Desarguesian affine spaces of dimension ≥ 2. One can easily obtain such an axiom system for spaces of dimension ≥ 3 by first rephrasing the axioms given by KUSAK [4] vor affine spaces of dimension ≥ 3 in terms of collinearity L, with $L(abc)$ to be read 'points a, b, c are collinear', instead of parallelism, then replacing every occurrence of $L(xyz)$ by $B(xyz) \vee B(yzx) \vee B(zxy)$, and adding order axioms, e. g. as in [8]. Let θ be the sentence obtained as the conjunction of all the axioms mentioned above. Let θ' be the conjunction of all the axioms for Desarguesian ordered affine planes from [11]. Then Δ may be chosen to be $\theta \vee \theta'$.
We further need
(a) axioms to ensure that $\equiv$ is a nondegenerate equivalence relation between segments, i. e. $ab \equiv ba$, $ab \equiv cd \wedge ab \equiv ef \rightarrow cd \equiv ef$, $aa \equiv bb$, $ab \equiv cc \rightarrow a = b$;
(b) a segment transport axiom, such as

$$(\forall abc)(\exists d)(\forall e)\,[B(cad) \wedge ab \equiv ad \wedge (a \neq c \wedge B(cae) \wedge ab \equiv ae \rightarrow d = e)];$$

(c) an axiom stating that the affinely defined midpoint of any segment ab (as the intersection point of ab with cd, where c and d are two different points with $ac \parallel bd$ and $ad \parallel bc$) is equidistant from a and b;
(d) an axiom stating that if $abcd$ is a parallelogram with $ab \parallel cd$ and $bc \parallel ad$, then $ab \equiv cd$ and $bc \equiv ad$;
(e) an axiom stating that one obtains an isosceles triangle by drawing a parallel to the base

[1] I know of no example of an equidistance preserving transformation between normed spaces with non-Archimedean ordered scalar fields that would not be an affine similarity, although the known proofs of VOGT's theorem depend heavily upon the Archimedeanity of the real field.

[2] One can easily translate our definitions into TC logic, where the fact that there is an explicit definition may be stronger, for I don't know whether our Preservation and Definability Theorem remains valid in TC logic.

of an isosceles triangle, i. e.

$$L(oab) \wedge L(oa'b') \wedge \neg L(oaa') \wedge b \neq o \wedge oa \equiv oa' \wedge aa' \parallel bb' \rightarrow ob \equiv ob';$$

(f) the weak (equality is allowed) triangle inequality, i. e. (the 'triangle' is abc, but notice that there is no condition of non-collinearity for these points)

$$(B(ba'c) \vee B(bca')) \wedge ba \equiv ba' \wedge B(ba'c') \wedge a'c' \equiv ac \rightarrow B(bcc');$$

(g) an axiom stating that there is a 'triangle' whose sides are three given segments that satisfy the weak triangle inequality;
(h) with the relation $\leq$ of inequality between the lengths of segments defined, as in Schnabel [9], by

$$ab \leq cd :\leftrightarrow (\forall m)(\exists s)\, [cm \equiv dm \rightarrow ab \equiv cs \wedge cm \equiv sm]$$

the axiom stating that any two segments are comparable under $\leq$, i. e. $ab \leq cd \vee cd \leq ab$;
(i) the Archimedean axiom, which is the only one whose expression requires infinitary logic, and which may be written, with $P(abcd)$ standing for a, b, c, d are the four vertices of a parallelogram, i. e. $ab \parallel cd$ and $bc \parallel ad$, as:

$$(\forall abx_1d)\, a \neq b \rightarrow \bigvee_{n=2}^{\infty} \{(\exists x_2 \dots x_n y_1 \dots y_{n-1})\, x_1x_2 \equiv ab \wedge (B(x_1x_2d) \vee B(x_1dx_2))$$
$$\bigwedge_{i=1}^{n-1} [P(x_ix_{i+1}y_{i+1}y_i) \wedge P(y_iy_{i+1}x_{i+2}x_{i+1})] \wedge B(x_1dx_n)\}.$$

Let Σ denote the axiom system consisting of Δ as well as of the axioms mentioned above. We can coordinatize models of Σ as usual and turn them into right vector spaces with Archimedean ordered fields as scalars (given that Archimedean ordered skew fields are commutative), with B having its usual analytic interpretation, and where to any vector $\mathbf{x}$ we may associate a *norm* $\|\mathbf{x}\|$, a non-negative element of the scalar skew field, such that $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = \mathbf{0}$, $\|\mathbf{x}\lambda\| = \|\mathbf{x}\| \cdot |\lambda|$, and $\|\mathbf{a} + \mathbf{b}\| \leq\ |\mathbf{a}\| +\ |\mathbf{b}\|$. For any vectors $\mathbf{a}$ and $\mathbf{b}$ we may thus define a *distance* by $d(\mathbf{a}, \mathbf{b}) := \|\mathbf{a} - \mathbf{b}\|$, and we have $ab \equiv cd$ iff $d(\mathbf{a}, \mathbf{b}) = d(\mathbf{c}, \mathbf{d})$.

3 Definitional understandings of Vogt's and Skof's theorems

We do not know whether the original variant of Vogt's theorem still holds for models of Σ, i. e. if $\equiv$-preserving surjections between models of Σ also preserve the betweenness relation B.
We shall prove the following theorem, the first part of which is the definitional counterpart of the weak variant of Vogt's theorem, the second part corresponding to Skof's variant of Vogt's theorem, for mappings that are not assumed surjective.

Theorem. *B and $\neq$ are definable in terms of $\equiv$, and B is definable by positive existential formulas in terms of $\equiv$ and $\neq$, the definitions being theorems of Σ.*

Proof. We first show how to define the relation $\equiv_2$in terms of $\equiv$, with $\mathbf{ab} \equiv_2 \mathbf{cd}$ to be interpreted as $d(\mathbf{a}, \mathbf{b}) = 2d(\mathbf{c}, \mathbf{d})$, by

$$ab \equiv_2 cd :\leftrightarrow (\exists e)\, ae \equiv cd \wedge be \equiv cd \wedge [(\forall xy)\,(xa \equiv xb \wedge ya \equiv yx) \to (\exists z)\, zc \equiv xy \wedge zd \equiv xy].$$

We can now define the midpoint relation. To this end, we recursively define a sequence of relations φ_n by[3]

$$\begin{aligned} \varphi_0(a,b,x) &:= xa \equiv xb \wedge ab \equiv_2 xa, \text{ and} \\ \varphi_{n+1}(a,b,x) :&\leftrightarrow \varphi_n(a,b,x) \wedge [(\forall x_3)(\exists x_1 x_2 y)\, \varphi_0(a,b,x_3) \\ &\to \bigwedge_{i=1}^{2} \varphi_0(a,b,x_i) \wedge xy \equiv_2 x_3 x \wedge xy \leq x_1 x_2] \end{aligned}$$

The definition of the midpoint relation M, with $M(\mathbf{abc})$ to be interpreted as $\mathbf{a} + \mathbf{c} = 2\mathbf{b}$ with $\mathbf{a} \neq \mathbf{c}$, is

$$M(abc) \leftrightarrow a \neq c \bigwedge_{n=0}^{\infty} \varphi_n(a,c,b).$$

Let (the conjuncts being considered to exist only insofar as the indices i of the corresponding x_i exist, i. e. if n is large enough):

$$\begin{aligned} \alpha_n(a,b,x_{n-1}) :&\leftrightarrow a \neq b \wedge (\exists x_1 \ldots x_{n-2})\, M(abx_1) \wedge M(bx_1x_2) \bigwedge_{i=2}^{n-2} M(x_{i-1}x_ix_{i+1}), \\ \beta_k(a,b,y_k) :&\leftrightarrow a \neq b \wedge (\exists y_1 \ldots y_{k_1})\, M(ay_1b) \bigwedge_{i=1}^{k-1} M(ay_{i+1}y_i). \end{aligned}$$

The two formulas are to be interpreted as: $\alpha_n(\mathbf{a}, \mathbf{b}, \mathbf{x})$ iff $\mathbf{a} \neq \mathbf{b}$ and $\mathbf{x}$ is a point on ray $\overrightarrow{\mathbf{ab}}$, such that $d(\mathbf{a}, \mathbf{x}) = n \cdot d(\mathbf{a}, \mathbf{b})$, and $\beta_k(\mathbf{a}, \mathbf{b}, \mathbf{y})$ iff $\mathbf{a} \neq \mathbf{b}$ and $\mathbf{y}$ is a point on segment $\mathbf{ab}$, such that $d(\mathbf{a}, \mathbf{y}) = 2^{-k} \cdot d(\mathbf{a}, \mathbf{b})$.
Let now

$$\psi_{n,k}(a,b,c,d) :\leftrightarrow a \neq b \wedge c \neq d \wedge (\exists euv)\, ce \equiv au \wedge de \equiv av \wedge \beta_k(a,b,v) \wedge \alpha_n(a,v,u).$$

The interpretation of $\psi_{n,k}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ is that $\mathbf{a}, \mathbf{b}$ as well as $\mathbf{c}, \mathbf{d}$ are different, and that there is a point $\mathbf{e}$ such that $d(\mathbf{c}, \mathbf{e}) = n2^{-k} \cdot d(\mathbf{a}, \mathbf{b})$ and $d(\mathbf{d}, \mathbf{e}) = 2^{-k} \cdot d(\mathbf{a}, \mathbf{b})$. By the triangle inequality this implies that[4]

$$\frac{(n-1) \cdot d(\mathbf{a}, \mathbf{b})}{2^k} \leq d(\mathbf{b}, \mathbf{c}) \leq \frac{(n+1) \cdot d(\mathbf{a}, \mathbf{b})}{2^k}.$$

We are now ready to define

$$\gamma(a,b,c) :\leftrightarrow \bigwedge_{k=1}^{\infty} \{ \bigvee_{n=1}^{\infty} [\psi_{n,k}(a,b,b,c) \wedge \psi_{n+2^k,k}(a,b,a,c)] \},$$

[3]For the rationale behind the φ_n's and their function in defining B cf. [1, §10.4].

[4]We are not pedantic about writing the scalar multipliers to the right, since all of them are rational scalars, and thus it is irelevant on which side they are written.

to be interpreted as '$\mathbf{a}, \mathbf{b}, \mathbf{c}$ are three different points, such that $d(\mathbf{a}, \mathbf{b}) + d(\mathbf{b}, \mathbf{c}) = d(\mathbf{a}, \mathbf{c})$'. We are finally ready to define B. Its definition is:

$$\begin{aligned} B(abc) \quad \leftrightarrow \quad & a = b \vee b = c \vee \{ \bigwedge_{n=1}^{\infty} [(\exists m_1 \dots m_{2^n-1}) \bigwedge_{i=2}^{2^n-2} M(m_{i-1} m_i m_{i+1}) \wedge M(a m_1 m_2) \\ & \wedge M(m_{2^n-2} m_{2^n-1} c) \wedge (\bigvee_{i=1}^{2^n-2} (\gamma(m_i, b, m_{i+1}) \vee \gamma(a, b, m_1) \vee \gamma(m_{2^n-1}, b, c))] \} \end{aligned}$$

We may now define $\neq$, by first defining

$$\delta(z_0, x, z_n) :\leftrightarrow (\exists z_1 \dots z_{n-1}) \bigwedge_{i=0}^{n-1} z_i z_{i+1} \equiv z_0 x,$$

with $\delta(\mathbf{a}, \mathbf{b}, \mathbf{c})$ to be interpreted as $d(\mathbf{a}, \mathbf{c}) \leq n \cdot d(\mathbf{a}, \mathbf{b})$. We have

$$x \neq y :\leftrightarrow (\forall z) \bigvee_{n=2}^{\infty} \delta(x, y, z).$$

□

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Department of Integrative Studies, Arizona State University West, P. O. Box 37100, Phoenix AZ 85069-7100, USA

E-mail: pamb@math.west.asu.edu